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Ricci Recurrent Space-Times with Torsion

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A Ricci recurrent space-time with covariantly constant stress tensor is an Einstein space-time. We extend this result to Ricci recurrent space-times with torsion. The result is applied to the case of Riemann–Cartan space-times with spin density.

1. INTRODUCTION

Recurrent spaces in Riemannian manifolds have been investigated in detail by several authors (Thompson, 1969; Ruse, 1950). Applications to general relativity have been given by Sciama (1961), who investigated recurrent radiation, and by Hall (1977), who applied recurrent Riemannian spaces to the study of Petrov and Sègre classification of gravitational fields. More recently recurrent Riemann–Cartan space-times have been investigated by Garcia de Andrade (1990) in connection with the Sègre classification of tensors in space-times with torsion (Garcia de Andrade, 1989). With the advent of theories with the propagation of torsion in the vacuum, such as study seems to be perfectly justifiable, although it is not possible in Einstein–Cartan–Sciama–Kibble theory. In this paper I present some algebraic results on Ricci recurrent spaces with torsion. The first theorem shows that a Ricci recurrent space-time which obeys the Einstein field equation has the structure of an Einstein space. The main result is an extension of this result to space-times with torsion. Finally, the result is applied to a spinning fluid in ECSK.

2. THEOREMS ON RICCI RECURRENT SPACES WITH TORSION

Let us consider a simple proof of the following theorem.

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Theorem 1. A Ricci recurrent Riemannian space-time where the Einstein field equations (where the energy-momentum tensor is covariantly constant in vacuum) are obeyed is an Einstein space.

Proof. Let us consider the Einstein field equations in the form

$$R_{ab} = \varkappa(T_{ab} - \tfrac{1}{2} g_{ab} T) \tag{2.1}$$

Now apply the Riemannian covariant derivative to both sides of expression (2.1) to obtain

$$R_{ab;e} - \tfrac{1}{2} g_{ab} R_{,e} = \varkappa T_{ab;e} \tag{2.2}$$

Using the definition of a Ricci recurrent space-time

$$R_{ab;e} = R_{ab} \xi_e \tag{2.3}$$

we obtain

$$R_{ab} \xi_e - \tfrac{1}{2} R g_{ab} \xi_e = \varkappa T_{ab;e} \tag{2.4}$$

There are two distinct cases here: The first is the one where the energy-momentum tensor is covariantly constant or

$$T_{ab;c} = 0 \tag{2.5}$$

The expression (2.4) reduces to

$$R_{ab} = \Lambda g_{ab} \tag{2.6}$$

where $\Lambda = \frac{1}{2} g_{ab} R$. This is exactly the form of an Einstein space. The second case is the contraction of the expression (2.4) as

$$R_{ab} \xi^b = \tfrac{1}{2} R \xi_a \tag{2.7}$$

which shows that the Ricci tensor has an eigenvalue proportional to the Ricci scalar, when the eigenvector is the current vector. The vacuum result is trivial, since then $T_{ab} = 0$ and (2.6) is immediate.

Now we formulate an extension of this theorem in the following form.

Theorem 2. In a Riemannian space-time with torsion (RC), where the Einstein–Cartan field equations are obeyed, the Ricci recurrent tensor is an Einstein space, in the case of a Weyssenhoff fluid, where the torsion obeys the constraint $S^c_{ab} = S_{ab} u^c$, or $S^c_{ab} = 0$, where S^c_{ab} is the spin density tensor connected to the torsion of space-time. The condition of Einstein space also imposes an additional constraint on the Riemann–Cartan tensor of the type $R_{abcd} u^d = 0$, where u^d is the four-velocity of the Weyssenhoff fluid.

Proof. The contracted Bianchi identity in Riemann–Cartan space-time is much more involved than the corresponding one in general relativity, or

$$\nabla_a T^a_b = S^c_{bd} R^d_c + \tfrac{1}{2} S^e_{ac} R^{ac}_{eb} + S_d (R^d_b - \tfrac{1}{2} R \delta^d_b) \tag{2.8}$$

where the last term vanishes because the trace of the torsion tensor vanishes in the case of a Weyssenhoff fluid. Making use of the Einstein-Cartan field equations on the right-hand side of expression (28), we obtain

$$\begin{aligned}\nabla_a(R^a_b - \tfrac{1}{2}R\delta^a_b) &= \nabla_a T^a_b = \nabla_a R^a_b - \tfrac{1}{2}\nabla_a R\delta^a_b \\ &= S^c_{bd}R^d_c + \tfrac{1}{2}S^e_{ac}R^{ac}_{eb} \end{aligned} \tag{2.9}$$

Applying now the Ricci recurrent condition in (2.9) yields

$$(R^a_b - \tfrac{1}{2}\delta^a_b R)\xi_a = \tfrac{1}{2}S^e_{ac}R^{ac}_{eb}u^b \tag{2.10}$$

where we have used $\xi^a = u^a$, u^a being the four-velocity of the fluid, and the application of the Weyssenhoff fluid conditions leads to (2.10). From expression (2.10) it is easy to see that the condition

$$R_{abcd}u^d = 0 \tag{2.11}$$

is equivalent to the Einstein space condition $R_{ab} = \Lambda g_{ab}$. Applications of the above theorem may prove useful in the study of the Petrov-type classification of gravitational fields in space-times with torsion, since the expression (2.11) appears very often in general relativity in connection with the Petrov type of gravitational fields. Equation (2.11) implies by contraction $R_{bd}u^d = 0$, which is also important for the Sègre classification of the Ricci tensor (Hall, 1983). Further investigations in this direction are now in progress.

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